A Bäcklund transformation for L-isothermic surfaces

2009 J. Phys. A: Math. Theor. 42 404015

(http://iopscience.iop.org/1751-8121/42/40/404015)

J. Phys. A: Math. Theor. **42** (2009) 404015 (12pp) doi:10.1088/1751-8113/42/40/404015

A Bäcklund transformation for L-isothermic surfaces

C Rogers[1,2] **and A Szereszewski**[3]

[1] Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
[2] Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems, School of Mathematics, The University of New South Wales, Sydney, NSW 2052, Australia
[3] Institute of Theoretical Physics, University of Warsaw, Hoża 69, 00-681 Warsaw, Poland

E-mail: colinr@maths.unsw.edu.au and aszer@fuw.edu.pl

Received 31 December 2008, in final form 2 January 2009
Published 16 September 2009
Online at stacks.iop.org/JPhysA/42/404015

Abstract
It is established that a Bäcklund transformation for L-isothermic surfaces is associated with a Darboux transformation for a non-homogeneous linear Schrödinger equation. A Lax pair for L-isothermic surfaces is presented and it is shown that a quartet of eigenfunctions contained therein may be explicitly represented in terms of linearly independent solutions of a linear Schrödinger equation with a potential involving the Bäcklund parameter. A permutability theorem is presented whereby L-isothermic surfaces may be constructed and the action of the Bäcklund transformation on a class of generalized Dupin cyclides is considered.

PACS numbers: 02.40.Hw, 02.30.Hq, 02.30.Jr

1. Introduction

The remarkable links that exist between the classical differential geometry of surfaces and modern soliton theory are well established (see e.g. [1]). The Bäcklund–Darboux transformations with their associated nonlinear superposition principles are notable in this connection [2]. Isothermic surfaces constitute an important sub-class of surfaces with a solitonic connection. They have been extensively studied by luminaries such as Bour [3], Darboux [4], Calapso [5] and Bianchi [6]. In more recent times, it has been established by Cieśliński *et al* [7, 8] that the classical Gauss–Mainardi–Codazzi system associated with isothermic surfaces is integrable in the modern solitonic sense. Indeed, a particular reduction of this isothermic system due to Calapso [5] may be shown to be linked to the zoomeron equation as set down in a solitonic context as a specialization of the matrix boomeron equation by Calogero and Degasperis [9–11]. The classical Bäcklund transformation for isothermic surfaces in $\mathbb{R}^3$ was originally set down by Darboux [4] and was subsequently discussed in detail by Bianchi [6]. Its formulation in terms of a matrix Darboux transformation is due to Cieśliński [7]. Bäcklund transformations and discrete analogues of isothermic surfaces in spaces of arbitrary dimension have been constructed by Schief [12]. Recently, in a study of integrable shell-membrane systems, a Láme-type equation was derived whereby a family of parallel L-isothermic surfaces could be constructed [13, 14]. It is remarked that parallel isothermic surfaces of the Dupin cyclide type arise naturally in liquid crystal theory [15] and are also of importance as offset geometries in computer-aided engineering design [16]. Here, we adopt the formulation of [14] to discuss the aspects of L-isothermic surfaces and, in particular, to generate L-isothermic surfaces via the action of a Bäcklund transformation on generalized Dupin cyclides.

2. Construction of L-isothermic surfaces via a linear Schrödinger equation

The Lie sphere geometry was introduced by Lie in 1872 [17]. Subsequently, important contributions to the Lie sphere geometry and its subgeometries (such as the Laguerre geometry) were made by Blaschke [18]. L-isothermic surfaces (surfaces with isothermic spherical representation) appear naturally in the context of Laguerre geometry. In recent years, Musso and Nicolodi studied the subject using Cartan's moving frame method [19–26].

In [14], parallel L-isothermic surfaces were constructed via solutions of a non-homogeneous linear Schrödinger equation. Thus, let Σ be a two-dimensional surface parametrized in terms of curvature coordinates (α, β) and $\mathbf{N}$ be a normal vector of Σ. The first and second fundamental forms are given by

$$\mathrm{I} = A_1^2\,\mathrm{d}\alpha^2 + A_2^2\,\mathrm{d}\beta^2, \tag{2.1}$$

$$\mathrm{II} = \kappa_1 A_1^2\,\mathrm{d}\alpha^2 + \kappa_2 A_2^2\,\mathrm{d}\beta^2, \tag{2.2}$$

where κ_1, κ_2 denote the principal curvatures. The condition that the surface Σ be L-isothermic is that its third fundamental form $\mathrm{III} = \mathrm{d}\mathbf{N}\cdot\mathrm{d}\mathbf{N}$[4] be conformally flat in (α, β). Under the assumption that $A_1\kappa_1 = A_2\kappa_2 = -e^{\theta}$, we obtain

$$\mathrm{III} = e^{2\theta}(\mathrm{d}\alpha^2 + \mathrm{d}\beta^2). \tag{2.3}$$

The method of construction of the family of parallel L-isothermic surfaces involves a complex potential $\mathsf{U}(z)$, where $z = \alpha + \mathrm{i}\beta$, and a real function $P(\alpha, \beta)$ obeying the Moutard-type equation

$$P_{\alpha\beta} = 2(\mathrm{Im}\,\mathsf{U})P. \tag{2.4}$$

The construction can be summarized as follows.

Proposition 1. *Let* $\mathsf{U}(z)$ *and* P *satisfy (2.4) and* T_0 *be a real solution of the non-homogeneous linear Schrödinger equation:*

$$T_{zz} + \mathsf{U}T = \tfrac{1}{4}P. \tag{2.5}$$

Then,

$$\mathbf{r} = e^{-\theta}b_z\mathbf{I} + e^{-\theta}b_{\bar z}\bar{\mathbf{I}} + b\mathbf{N}, \tag{2.6}$$

where

$$e^{-\theta} = \frac{1}{2}(|\Phi_1|^2 + |\Phi_2|^2), \tag{2.7}$$

[4] The dot $(\cdot)$ denotes the scalar product in the Euclidean space.

$$b = \frac{2T_0}{|\Phi_1|^2 + |\Phi_2|^2} + \mathfrak{b}, \qquad \mathfrak{b} \in \mathbb{R}, \tag{2.8}$$

is the position vector of an L-isothermic surface. The unit tangent vectors **X**, **Y** *and normal vector* **N** *of the surface are expressed in terms of* Φ_1 *and* Φ_2*, which are linearly independent solutions of a homogeneous version of (2.5) with the unit Wronskian, via the relations*

$$\begin{aligned} \mathbf{I} = \mathbf{X} + i\mathbf{Y} &= \frac{1}{|\Phi_1|^2 + |\Phi_2|^2} \begin{pmatrix} \Phi_2^2 - \Phi_1^2 \\ \mathrm{i}\left(\Phi_1^2 + \Phi_2^2\right) \\ 2\Phi_1\Phi_2 \end{pmatrix}, \\ \mathbf{N} &= -\frac{1}{|\Phi_1|^2 + |\Phi_2|^2} \begin{pmatrix} \Phi_1\bar{\Phi}_2 + \bar{\Phi}_1\Phi_2 \\ \mathrm{i}(\bar{\Phi}_1\Phi_2 - \Phi_1\bar{\Phi}_2) \\ |\Phi_1|^2 - |\Phi_2|^2 \end{pmatrix}. \end{aligned} \tag{2.9}$$

The coefficients of the first fundamental form (2.1) can be calculated from

$$P = A_1 - A_2, \qquad R = A_1 + A_2, \tag{2.10}$$

where

$$R = 4e^{-\theta} b_{z\bar{z}} + 2e^{\theta} b. \tag{2.11}$$

If the position vector is known, then the potential U is given by

$$\mathsf{U} = -e^{\theta}(e^{-\theta})_{zz} \tag{2.12}$$

and Φ_1, Φ_2 may be found via the relations

$$\Phi_1^2 = -e^{-\theta}(I_1 + \mathrm{i}\, I_2), \tag{2.13}$$

$$\Phi_2^2 = e^{-\theta}(I_1 - \mathrm{i}\, I_2), \tag{2.14}$$

where $\mathbf{I} = (I_1, I_2, I_3)^T$.[5]

It has been shown in [27] that the above approach allows the construction of a Weierstrass-type representation of surfaces which are both L-isothermic and L-minimal. The position vector of such surfaces is given by [28]

$$\mathbf{r}_L = \mathrm{Re} \begin{pmatrix} \int(-m_1 + (m_2 - \bar{m}_2)\rho + m_3\rho^2)F(\rho)\,\mathrm{d}\rho \\ \mathrm{i}\int(m_1 + (m_2 + \bar{m}_2)\rho + m_3\rho^2)F(\rho)\,\mathrm{d}\rho \\ \int(m_2 + (m_1 + m_3)\rho + \bar{m}_2\rho^2)F(\rho)\,\mathrm{d}\rho \end{pmatrix} + \frac{\mathcal{H}/\mathcal{K}}{1 + \rho\bar{\rho}} \begin{pmatrix} \rho + \bar{\rho} \\ \mathrm{i}(\rho - \bar{\rho}) \\ 1 - \rho\bar{\rho} \end{pmatrix}. \tag{2.15}$$

where $m_1, m_3 \in \mathbb{R}$, $m_2 \in \mathbb{C}$,

$$\frac{\mathcal{H}}{\mathcal{K}} = -\mathrm{Re}\int(m_2 - (m_1 - m_3)\rho - \bar{m}_2\rho^2)F(\rho)\,\mathrm{d}\rho + \mu, \qquad \mu \in \mathbb{R}, \tag{2.16}$$

and $F(\rho)$ is an arbitrary holomorphic function of ρ. The functions $\mathcal{H}$ and $\mathcal{K}$ denote the mean and Gauss curvature, respectively. The description of the L-isothermic surfaces via the potential equation (2.5) proves geometrically convenient and, in particular, appropriate transformations of T_0, and Φ_1 and Φ_2 correspond to Laguerre transformations [27].

[5] The sign of $\Phi_1\Phi_2$ can be recovered from I_3.

3. Bäcklund and Darboux transformations

The Bäcklund transformations for L-isothermic surfaces have been studied both by Bianchi [29] and Eisenhart [30]. The basic result is as follows.

Proposition 2 (A Bäcklund transformation for L-isothermic surfaces)**.** *Let* $\mathbf{r}$ *be the position vector of an L-isothermic surface* Σ*. Then, a second L-isothermic surface* $\tilde{\Sigma}$ *is given by*

$$\tilde{\mathbf{r}} = \mathbf{r} - \frac{\lambda}{m\sigma t}(\mu\mathbf{X} + \nu\mathbf{Y} + \sigma\mathbf{N}), \tag{3.1}$$

where m *is a real 'Bäcklund parameter' and* $\lambda, \sigma, t, \mu, \nu$ *are 'eigenfunctions' of the compatible linear system:*

$$\begin{pmatrix}\lambda\\ \sigma\\ t\\ \mu\\ \nu\end{pmatrix}_\alpha = \begin{pmatrix}0 & 0 & 0 & A_1 & 0\\ 0 & 0 & 0 & e^\theta & 0\\ 0 & 0 & 0 & e^{-\theta} & 0\\ 0 & me^{-\theta} - e^\theta & me^\theta & 0 & -\theta_\beta\\ 0 & 0 & 0 & \theta_\beta & 0\end{pmatrix}\begin{pmatrix}\lambda\\ \sigma\\ t\\ \mu\\ \nu\end{pmatrix}, \tag{3.2}$$

$$\begin{pmatrix}\lambda\\ \sigma\\ t\\ \mu\\ \nu\end{pmatrix}_\beta = \begin{pmatrix}0 & 0 & 0 & 0 & A_2\\ 0 & 0 & 0 & 0 & e^\theta\\ 0 & 0 & 0 & 0 & -e^{-\theta}\\ 0 & 0 & 0 & 0 & \theta_\alpha\\ 0 & -me^{-\theta} - e^\theta & me^\theta & -\theta_\alpha & 0\end{pmatrix}\begin{pmatrix}\lambda\\ \sigma\\ t\\ \mu\\ \nu\end{pmatrix}, \tag{3.3}$$

which satisfy the admissible constraint

$$\mu^2 + \nu^2 + \sigma^2 = 2m\sigma t. \tag{3.4}$$

The transformation of the geometric quantities of Σ *reads as*

$$\tilde{A}_1 = -A_1 + \lambda\left(\frac{e^\theta}{\sigma} + \frac{e^{-\theta}}{t}\right), \tag{3.5}$$

$$\tilde{A}_2 = A_2 - \lambda\left(\frac{e^\theta}{\sigma} - \frac{e^{-\theta}}{t}\right), \tag{3.6}$$

$$e^{\tilde{\theta}} = \frac{\sigma}{t}e^{-\theta}. \tag{3.7}$$

Here, this Bäcklund transformation is connected with the construction of L-isothermic surfaces via the potential U of the linear Schrödinger equation (2.5) by the following result.

Proposition 3. *Let the complex function* U *be a potential associated with the L-isothermic surfaces* Σ*. The Bäcklund transformation of* Σ *corresponds to the Darboux transformation of potential* U*, namely*

$$\tilde{\mathsf{U}} = \mathsf{U} + 2(\log\hat{\sigma})_{zz}, \tag{3.8}$$

where the real function $\hat{\sigma} = 2e^{-\theta}\sigma$ *satisfies*

$$\hat{\sigma}_{zz} + \mathsf{U}\hat{\sigma} = \frac{m}{2}\hat{\sigma}. \tag{3.9}$$

The transformed solutions of $\Phi_{zz} + \mathsf{U}\Phi = 0$ *read as*

$$\tilde{\Phi}_1 = \sqrt{\frac{2}{|m|}}(\Phi_{1z} - (\log\hat{\sigma})_z\Phi_1)\,\mathrm{sgn}(m),$$
$$\tilde{\Phi}_2 = \sqrt{\frac{2}{|m|}}(\Phi_{2z} - (\log\hat{\sigma})_z\Phi_2). \tag{3.10}$$

Proof. It is readily shown that if σ is a solution of the system (3.2)–(3.3) then $\hat{\sigma}$, defined in proposition 3, satisfies the relations

$$\hat{\sigma}_{\alpha\alpha} - \hat{\sigma}_{\beta\beta} + e^{\theta}((e^{-\theta})_{\beta\beta} - (e^{-\theta})_{\alpha\alpha})\hat{\sigma} = 2m\hat{\sigma}, \tag{3.11}$$

$$\hat{\sigma}_{\alpha\beta} = e^{\theta}(e^{-\theta})_{\alpha\beta}\hat{\sigma}. \tag{3.12}$$

Hence, if $z = \alpha + \mathrm{i}\beta$, then (3.11) and (3.12) combine to produce (3.9). The transformed potential $\tilde{\mathsf{U}}$ is defined by

$$\tilde{\mathsf{U}} = -e^{\tilde{\theta}}(e^{-\tilde{\theta}})_{zz} \tag{3.13}$$

and using (3.7) together with (3.2) and (3.3), it is verified that $\tilde{\mathsf{U}}$ satisfies relation (3.8). Formulae (3.10) constitute standard transforms of solutions associated with a Darboux transformation. □

It turns out that solutions σ, t, μ, ν of the system (3.2)–(3.3) can be conveniently expressed in terms of Φ_1, Φ_2 and linearly independent solutions φ_1, φ_2 for

$$\varphi_{zz} + \left(\mathsf{U} - \frac{m}{2}\right)\varphi = 0. \tag{3.14}$$

This result is incorporated in the following.

Proposition 4. *The solution* $\{\sigma, t, \mu, \nu\}$ *of the system (3.2)–(3.3) is given by*

$$\sigma = \frac{1}{2}|S|^2 e^{\theta}, \tag{3.15}$$

$$t = \frac{1}{2m}|S|^4\left(\frac{1}{\sigma}\right)_{z\bar{z}}, \tag{3.16}$$

$$\mu = e^{-\theta}\sigma_{\alpha}, \tag{3.17}$$

$$\nu = e^{-\theta}\sigma_{\beta}, \tag{3.18}$$

where

$$S = s_1\varphi_1 + s_2\varphi_2, \qquad s_1, s_2 \in \mathbb{C}, \tag{3.19}$$

and functions φ_1, φ_2 *are two linearly independent solutions for (3.14) while* θ *is defined in (2.7).*

Proof. In terms of $\hat{\sigma}$, as defined in proposition 3, the quadratic constraint (3.4) can be rewritten as

$$\mu^2 + \nu^2 + \sigma^2 - 2m\sigma t = -2e^{-2\theta}\sigma^2(\log|\hat{\sigma}|)_{z\bar{z}} = 0. \tag{3.20}$$

Hence[6],

$$\hat{\sigma} = S(z)\overline{S(z)}, \tag{3.21}$$

[6] $\hat{\sigma}$ is chosen to be positive without loss of generality

where $S(z)$ is a holomorphic function of z. Moreover, the real function $\hat{\sigma}$ satisfies (3.14). Therefore,

$$S = s_1\varphi_1 + s_2\varphi_2, \qquad s_1, s_2 \in \mathbb{C}, \tag{3.22}$$

where φ_1, φ_2 are two linearly independent solutions of (3.14). Accordingly, function σ is given by (3.15) and straightforward calculation shows that (3.16)–(3.18) satisfy the system (3.2)–(3.3). □

It is noted that the equations for the eigenfunction λ, namely

$$\lambda_\alpha = A_1\mu, \qquad \lambda_\beta = A_2\nu, \tag{3.23}$$

can be treated separately from those for σ, t, μ, ν (cf (3.2) and (3.3)). Explicit integration for λ requires knowledge of the first fundamental form (2.1). It is also remarked that solution (3.16) for t can be rewritten in terms of transformed functions (3.10) in the following way:

$$t = \tfrac{1}{4}(|S\tilde{\Phi}_1|^2 + |S\tilde{\Phi}_2|^2). \tag{3.24}$$

4. A permutability theorem

Let $\mathbf{r}$ be a position vector of an L-isothermic surface and $\mathbf{r}_1$ and $\mathbf{r}_2$ be two Bäcklund transforms of $\mathbf{r}$ via $\mathbb{B}_{m_1}$ and $\mathbb{B}_{m_2}$, respectively. The following permutability theorem allows construction of a new L-isothermic surface from $\mathbf{r}_1$ and $\mathbf{r}_2$ in a purely algebraic manner.

Proposition 5. *If $\mathbf{r}_1$ and $\mathbf{r}_2$ are two Bäcklund transforms of $\mathbf{r}$, then*

$$\mathbf{R} = \frac{\begin{vmatrix} \mathbf{r} & \lambda_1 & \lambda_2 \\ \mathbf{r}_1 & j_1 & j_3 \\ \mathbf{r}_2 & j_4 & j_2 \end{vmatrix}}{\begin{vmatrix} j_1 & j_3 \\ j_4 & j_2 \end{vmatrix}} \tag{4.1}$$

gives the position vector of a new L-isothermic surface, where

$$\mathbf{r}_1 = \mu_1\mathbf{X} + \nu_1\mathbf{Y} + \sigma_1\mathbf{N}, \qquad j_1 = m_1\sigma_1 t_1, \tag{4.2}$$

$$\mathbf{r}_2 = \mu_2\mathbf{X} + \nu_2\mathbf{Y} + \sigma_2\mathbf{N}, \qquad j_2 = m_2\sigma_2 t_2, \tag{4.3}$$

$$j_3 = \frac{m_2}{m_2 - m_1}\left(\sigma_1\sigma_2 + \mu_1\mu_2 + \nu_1\nu_2 - m_1(\sigma_1 t_2 + \sigma_2 t_1)\right), \tag{4.4}$$

$$j_4 = \frac{m_1}{m_1 - m_2}\left(\sigma_1\sigma_2 + \mu_1\mu_2 + \nu_1\nu_2 - m_2(\sigma_1 t_2 + \sigma_2 t_1)\right). \tag{4.5}$$

and $\{\lambda_1, \sigma_1, t_1, \mu_1, \nu_1\}$ and $\{\lambda_2, \sigma_2, t_2, \mu_2, \nu_2\}$ are two sets of solutions for the system (3.2)–(3.3) for m_1 and $m_2 \neq m_1$ respectively. The potential $\mathsf{U}_\mathbf{R}$ which corresponds to $\mathbf{R}$ reads as

$$\mathsf{U}_\mathbf{R} = \mathsf{U} + 2\partial_{zz}\log\left(S_1 S_{2z} - S_{1z}S_2\right), \tag{4.6}$$

where S_1 and S_2 satisfy

$$S_{1zz} + \left(\mathsf{U} - \frac{m_1}{2}\right) S_1 = 0, \tag{4.7}$$

$$S_{2zz} + \left(\mathsf{U} - \frac{m_2}{2}\right) S_2 = 0. \tag{4.8}$$

Proof. A straightforward calculation shows that (4.1) is the L-isothermic surface corresponding to $\theta_{\mathbf{R}}$ given by

$$e^{-\theta_{\mathbf{R}}} = e^{-\theta}\left(1 + \frac{(m_2 - m_1)(\sigma_1 t_2 - \sigma_2 t_1)}{\sigma_1\sigma_2 + \mu_1\mu_2 + \nu_1\nu_2 - m_1\sigma_2 t_1 - m_2\sigma_1 t_2}\right). \tag{4.9}$$

The associated potential $\mathsf{U}_{\mathbf{R}} = -e^{\theta_{\mathbf{R}}}(e^{-\theta_{\mathbf{R}}})_{zz}$ is given by

$$\mathsf{U}_{\mathbf{R}} = \mathsf{U} + 2\partial_{zz}\log\left(\sigma_1\sigma_2 + \mu_1\mu_2 + \nu_1\nu_2 - m_1\sigma_2 t_1 - m_2\sigma_1 t_2\right), \tag{4.10}$$

where on using (3.2) and (3.3), it is seen that

$$\sigma_1\sigma_2 + \mu_1\mu_2 + \nu_1\nu_2 - m_1\sigma_2 t_1 - m_2\sigma_1 t_2 = \tfrac{1}{2}(\hat\sigma_{1z}\hat\sigma_{2\bar z} + \hat\sigma_{1\bar z}\hat\sigma_{2z} - \hat\sigma_2\hat\sigma_{1z\bar z} - \hat\sigma_1\hat\sigma_{2z\bar z}) \tag{4.11}$$

$$= \tfrac{1}{2}(S_1 S_{2z} - S_2 S_{1z})(\bar S_2\bar S_{1\bar z} - \bar S_1\bar S_{2\bar z}). \tag{4.12}$$

Here, we have used the fact that $\hat\sigma_1$ and $\hat\sigma_2$ satisfy (3.9) with m_1 and m_2 respectively. □

It is remarked that the function θ obeys the Liouville equation. Indeed, this is the Gauss equation of the L-isothermic surface. Thus, $\theta_{\mathbf{R}}$ likewise satisfies a Liouville equation

$$\Delta\theta_{\mathbf{R}} + e^{2\theta_{\mathbf{R}}} = 0, \tag{4.13}$$

where $\Delta = \partial_{\alpha\alpha} + \partial_{\beta\beta}$.

5. Illustration

Here, by way of illustration, we consider the action of the Bäcklund transformation on the generalized Dupin cyclides introduced in [14, 28]. These are L-isothermic canal surfaces:

$$\mathbf{r} = e^{\theta}\begin{pmatrix} (c_0F_1 - \mathfrak{b})\sin\alpha \\ -(c_0F_1 - \mathfrak{b})\sinh\beta \\ F_1\cosh\beta + \mathfrak{b}(c_0\cosh\beta - a_0\cos\alpha)\end{pmatrix} - \begin{pmatrix} F_2 \\ 0 \\ 0\end{pmatrix}, \tag{5.1}$$

where $a_0, c_0, \mathfrak{b}$ are real constants:

$$e^{-\theta} = a_0\cosh\beta - c_0\cos\alpha, \tag{5.2}$$

$$F_1(\alpha) = \int P(\alpha)\sin\alpha\,\mathrm{d}\alpha, \tag{5.3}$$

$$F_2(\alpha) = \int P(\alpha)\cos\alpha\,\mathrm{d}\alpha, \tag{5.4}$$

$P(\alpha)$ is an arbitrary function and

$$a_0^2 - c_0^2 = 1. \tag{5.5}$$

The surface (5.1) can be constructed via the method described in proposition 1 with the specializations

$$\mathsf{U} = \tfrac{1}{4}, \qquad P = P(\alpha). \tag{5.6}$$

The relevant geometric quantities for (5.1) are set down in appendix A. From propositions 2 and 4, the Bäcklund transformation of the surface (5.1) is given by (3.1) where the functions σ, t, μ and ν are defined in terms of linearly independent solutions of

$$\varphi_{zz} + \left(\frac{1}{4} - \frac{m}{2}\right)\varphi = 0. \tag{5.7}$$

There are three cases depending on the value of constant m.

Case 1. $m < \frac{1}{2}, \quad m \neq 0, \quad k = \sqrt{1-2m}$

In this case, $\varphi_1 = \cos\left(\frac{kz}{2}\right)$, $\varphi_2 = \sin\left(\frac{kz}{2}\right)$ and, in a generic case,

$$|S|^2 = 2\left(\cosh(k\beta + \beta_0) \pm \cos(k\alpha + \alpha_0)\right), \tag{5.8}$$

where $\alpha_0, \beta_0 \in \mathbb{R}$. The irrelevant constant factor on the right-hand side of (5.8) has been omitted. Hence,

$$\sigma = \frac{\cosh(k\beta + \beta_0) - \epsilon\cos(k\alpha + \alpha_0)}{a_0\cosh\beta - c_0\cos\alpha}, \tag{5.9}$$

$$\mu = \epsilon k\sin(k\alpha + \alpha_0) - c_0\sigma\sin\alpha, \tag{5.10}$$

$$\nu = k\sinh(k\beta + \beta_0) - a_0\sigma\sinh\beta, \tag{5.11}$$

$$t = a_0\left(\frac{1+k^2}{1-k^2}\cosh\beta\cosh(k\beta+\beta_0) - \frac{2k}{1-k^2}\sinh\beta\sinh(k\beta+\beta_0) - \epsilon\cosh\beta\cos(k\alpha+\alpha_0)\right)$$
$$+ c_0\left(\frac{\epsilon(1+k^2)}{k^2-1}\cos\alpha\cos(k\alpha+\alpha_0) + \frac{2\epsilon k}{k^2-1}\sin\alpha\sin(k\alpha+\alpha_0) + \cos\alpha\cosh(k\beta+\beta_0)\right), \tag{5.12}$$

where $\alpha_0, \beta_0 \in \mathbb{R}$ are constants of integration and $\epsilon = \pm 1$. Equations (3.23) may be integrated to obtain

$$\lambda = -(c_0F_1 - \mathfrak{b})\sigma + \epsilon k\int P(\alpha)\sin(k\alpha + \alpha_0)\,\mathrm{d}\alpha. \tag{5.13}$$

In the special case, the solution adopts the form

$$\sigma = \frac{e^{\epsilon k\beta}}{a_0\cosh\beta - c_0\cos\alpha}, \tag{5.14}$$

$$\mu = -c_0\sigma\sin\alpha, \tag{5.15}$$

$$\nu = \epsilon k e^{\epsilon k\beta} - a_0\sigma\sinh\beta, \tag{5.16}$$

$$t = \frac{a_0e^{\epsilon k\beta}}{1-k^2}((1+k^2)\cosh\beta - 2\epsilon k\sinh\beta) + c_0e^{\epsilon k\beta}\cos\alpha, \tag{5.17}$$

where $\epsilon = \pm 1$ and

$$\lambda = -(c_0F_1 - \mathfrak{b})\sigma + \lambda_0, \qquad \lambda_0 = \text{const}. \tag{5.18}$$

Case 2. $m = \frac{1}{2}$

In this case, $\varphi_1 = 1$, $\varphi_2 = z$ and, in a generic case,

$$|S|^2 = 2((\alpha + \alpha_0)^2 + (\beta + \beta_0)^2), \tag{5.19}$$

where $\alpha_0, \beta_0 \in \mathbb{R}$. The irrelevant constant factor on the right-hand side of (5.19) has been omitted. Hence,

$$\sigma = \frac{(\alpha+\alpha_0)^2 + (\beta+\beta_0)^2}{a_0\cosh\beta - c_0\cos\alpha}, \tag{5.20}$$

$$\mu = 2(\alpha + \alpha_0) - c_0\sigma\sin\alpha, \tag{5.21}$$

$$\nu = 2(\beta + \beta_0) - a_0\sigma\sinh\beta, \tag{5.22}$$

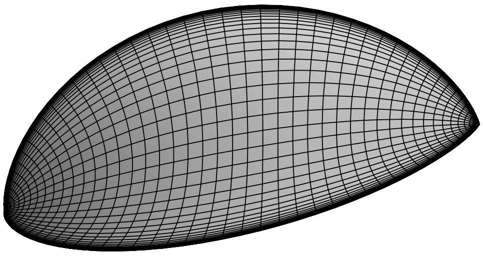

Figure 1. A surface for $c_0 = \frac{3}{2}$ defined by (5.31).

$$t = a_0[((\alpha+\alpha_0)^2 + (\beta+\beta_0)^2 + 4)\cosh\beta - 4(\beta+\beta_0)\sinh\beta] \tag{5.23}$$

$$+ c_0[((\alpha+\alpha_0)^2 + (\beta+\beta_0)^2 - 4)\cos\alpha - 4(\alpha+\alpha_0)\sin\alpha], \tag{5.24}$$

where $\alpha_0, \beta_0 \in \mathbb{R}$ are constants of integration. Equations (3.23) may be integrated to obtain

$$\lambda = -(c_0F_1 - \mathfrak{b})\sigma + 2\int(\alpha+\alpha_0)P(\alpha)\mathrm{d}\alpha. \tag{5.25}$$

In the special case, the solution adopts the form

$$\sigma = \frac{1}{a_0\cosh\beta - c_0\cos\alpha}, \tag{5.26}$$

$$\mu = -c_0\sigma\sin\alpha, \tag{5.27}$$

$$\nu = -a_0\sigma\sinh\beta, \tag{5.28}$$

$$t = a_0\cosh\beta + c_0\cos\alpha, \tag{5.29}$$

$$\lambda = -(c_0F_1 - \mathfrak{b})\sigma + \lambda_0, \qquad \lambda_0 = \text{const}, \tag{5.30}$$

and a new surface is likewise the generalized Dupin cyclide.

Case 3. $m > \frac{1}{2}, k = \sqrt{2m-1}$

The solution of the system (3.16)–(3.18) is given in appendix B.

Here, we consider the action of the Bäcklund transformation on the surface which is both L-isothermic and L-minimal, namely

$$\mathbf{r} = \frac{1}{2(a_0\cosh\beta - c_0\cos\alpha)}\begin{pmatrix} -a_0\sin\alpha\cos\alpha\cosh\beta \\ c_0\cos^2\alpha\sinh\beta \\ -\cos^2\alpha\cosh\beta \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -2\alpha \\ 0 \\ a_0 \end{pmatrix}, \tag{5.31}$$

$$\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \qquad -\infty < \beta < \infty.$$

The latter surface for $c_0 = \frac{3}{2}$ is displayed in figure 1. Figures 2 and 3 illustrate the action of the Bäcklund transformation on (5.31) for the two cases: $m = \frac{3}{8}$ and $m = \frac{1}{2}$. It is interesting to note that the original seed surface (5.31) fits precisely into its Bäcklund transform (vide figure 4).

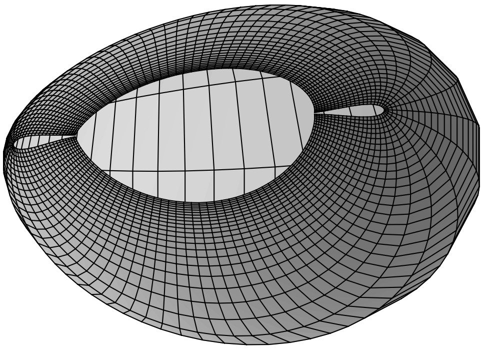

Figure 2. A Bäcklund transformation of the surface (5.31) with $c_0 = \frac{3}{2}$ for $m = \frac{3}{8}, \epsilon = 1, \alpha_0 = \frac{\pi}{2}, \beta_0 = 0$.

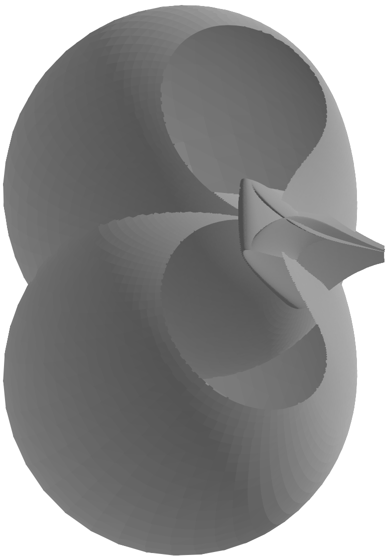

Figure 3. A Bäcklund transformation of the surface (5.31) with $c_0 = \frac{3}{2}$ for $m = \frac{1}{2}, \alpha_0 = -\frac{3}{2}\pi, \beta_0 = 0$.

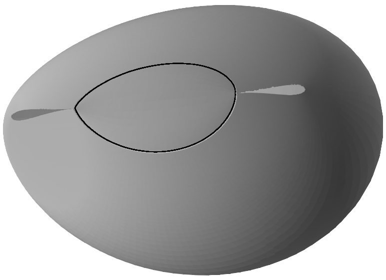

Figure 4. A seed surface (5.31) and its Bäcklund transform from figure 2.

Appendix A. The geometric quantities for generalized Dupin cyclides

The first, second and third fundamental forms for L-isothermic surfaces (5.1) read as

$$\mathrm{I} = (P(\alpha) - e^{\theta}(c_0 F_1 - \mathfrak{b}))^2\,\mathrm{d}\alpha^2 + e^{2\theta}(c_0 F_1 - \mathfrak{b})^2\,\mathrm{d}\beta^2, \tag{A.1}$$

$$\mathrm{II} = e^{\theta}(e^{\theta}(c_0 F_1 - \mathfrak{b}) - P(\alpha))\,\mathrm{d}\alpha^2 + e^{2\theta}(c_0 F_1 - \mathfrak{b})\,\mathrm{d}\beta^2, \tag{A.2}$$

$$\mathrm{III} = e^{2\theta}(\mathrm{d}\alpha^2 + \mathrm{d}\beta^2), \tag{A.3}$$

where e^{θ} and F_1 are defined in (5.2) and (5.3) respectively. The tangent vectors $\mathbf{X}$, $\mathbf{Y}$ and the normal vector $\mathbf{N}$ are given by

$$\mathbf{X} = e^{\theta}\begin{pmatrix} c_0 - a_0\cos\alpha\cosh\beta \\ -c_0\sin\alpha\sinh\beta \\ \sin\alpha\cosh\beta \end{pmatrix}, \qquad \mathbf{Y} = e^{\theta}\begin{pmatrix} a_0\sin\alpha\sinh\beta \\ a_0 - c_0\cos\alpha\cosh\beta \\ \cos\alpha\sinh\beta \end{pmatrix}, \tag{A.4}$$

$$\mathbf{N} = -e^{\theta}\begin{pmatrix} \sin\alpha \\ -\sinh\beta \\ a_0\cos\alpha - c_0\cosh\beta \end{pmatrix}. \tag{A.5}$$

Appendix B

The solution of the system (3.16)–(3.18) for generalized Dupin cyclides (5.1) in the case $m > \frac{1}{2}$ is given in a generic case by ($\epsilon = \pm 1$):

$$\sigma = \frac{\cosh(k\alpha + \alpha_0) - \epsilon\cos(k\beta + \beta_0)}{a_0\cosh\beta - c_0\cos\alpha},$$

$$\mu = k\sinh(k\alpha + \alpha_0) - c_0\sigma\sin\alpha,$$

$$\nu = \epsilon k\sin(k\beta + \beta_0) - a_0\sigma\sinh\beta,$$

$$t = a_0\left(\frac{\epsilon(k^2 - 1)}{1 + k^2}\cosh\beta\cos(k\beta + \beta_0) - \frac{2\epsilon k}{1 + k^2}\sinh\beta\sin(k\beta + \beta_0) + \cosh\beta\cos(k\alpha + \alpha_0)\right)$$
$$+ c_0\left(\frac{1 - k^2}{k^2 + 1}\cos\alpha\cosh(k\alpha + \alpha_0) - \frac{2k}{k^2 + 1}\sin\alpha\sinh(k\alpha + \alpha_0) - \epsilon\cos\alpha\cosh(k\beta + \beta_0)\right),$$

$$\lambda = -(c_0 F_1 - \mathfrak{b})\,\sigma + k\int P(\alpha)\sinh(k\alpha + \alpha_0)\,\mathrm{d}\alpha.$$

In the special case, the solution adopts the form

$$\sigma = \frac{e^{\epsilon k\alpha}}{a_0\cosh\beta - c_0\cos\alpha}, \tag{B.1}$$

$$\mu = \epsilon k e^{\epsilon k\alpha} - c_0\sigma\sin\alpha, \tag{B.2}$$

$$\nu = -a_0\sigma\sinh\beta, \tag{B.3}$$

$$t = a_0 e^{\epsilon k\alpha}\cosh\beta + \frac{c_0 e^{\epsilon k\alpha}}{1 + k^2}((1 - k^2)\cos\alpha - 2\epsilon k\sin\alpha), \tag{B.4}$$

$$\lambda = -(c_0 F_1 - \mathfrak{b})\sigma + \epsilon k\int P(\alpha)e^{\epsilon k\alpha}\,\mathrm{d}\alpha, \tag{B.5}$$

where $\epsilon = \pm 1$. Here, a new surface is the generalized Dupin cyclide with

$$\tilde{P}(\alpha) = -P(\alpha) + 2\epsilon k e^{-\epsilon k\alpha}\int P(\alpha)e^{\epsilon k\alpha}\,\mathrm{d}\alpha. \tag{B.6}$$

References

[1] Rogers C and Schief W K 2002 *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory* (*Cambridge Texts in Applied Mathematics*) (Cambridge: Cambridge University Press)

[2] Rogers C and Shadwick W F 1982 *Bäcklund Transformations and Their Applications* (New York: Academic)

[3] Bour E 1862 Theorie de la deformation des surfaces *J. l'Ecole Imperiale Polytech.* **19** (Cahier 39) 1–48

[4] Darboux G 1899 Sur les surfaces isothermiques *C. R. Acad. Sci.* **128** 1299–305

[5] Calapso P 1903 Sulla superficie a linee di curvatura isoterme *Rend. Circ. Mat. Palermo* **17** 275–86

[6] Bianchi L 1904 Ricerche sulle superficie isoterme e sulla deformazione delle quadriche *Ann. Mat.* **11** 93–157

[7] Cieśliński J 1997 The Darboux–Bianchi transformation for isothermic surfaces. Classical results versus the solitonic approach *Diff. Geom. Appl.* **7** 1–28

[8] Cieśliński J, Goldstein P and Sym A 1995 Isothermic surfaces in $\mathbb{E}^3$ as soliton surfaces *Phys. Lett.* A **205** 37–43

[9] Calogero F and Degasperis A 1976 Coupled nonlinear evolution equations solvable via the inverse spectral transform, and solitons that come back: the boomeron *Lett. Nuovo Cimento* **16** 425–33

[10] Calogero F and Degasperis A 1976 A Bäcklund transformations, nonlinear superposition principle, multisoliton solutions and conserved quantities for the 'boomeron' nonlinear evolution equation *Lett. Nuovo Cimento* **16** 434–8

[11] Degasperis A, Rogers C and Schief W K 2002 Isothermic surfaces generated via Darboux–Bäcklund transformations boomeron and zoomeron connections *Stud. Appl. Math.* **109** 39–65

[12] Schief W K 2001 Isothermic surfaces in spaces of arbitrary dimension: integrability, discretization and Bäcklund transformations. A discrete Calapso equation *Stud. Appl. Math.* **106** 85–137

[13] Rogers C and Schief W K 2003 On the equilibrium of shell membranes under normal loading. Hidden integrability *Proc. R. Soc.* A **459** 2449–62

[14] Schief W K, Szereszewski A and Rogers C 2007 The Lamé equation in shell membrane theory *J. Math. Phys.* **48** 073510

[15] Schief W K, Kléman M and Rogers C 2005 On a nonlinear elastic shell system in liquid crystal theory: generalized Willmore surface and Dupin cyclides *Proc. R. Soc.* A **461** 2817–37

[16] Martin R R, de Pont J and Sharock T J 1986 Cyclide surfaces in computer aided design *The Mathematics of Surfaces* ed J A Gregory (Oxford: Oxford University Press)

[17] Lie S 1872 Ueber Complexe, insbesondere Linien- und Kugel-Complexe, mit Anwendung auf die Theorie partieller Differential-Gleichungen *Clebsch Ann.* **V** 145–256

[18] Blaschke W 1929 *Vorlesungen über Differentialgeometrie und Geometrische Grundlagen von Einsteins Relativitätstheorie: III. Differentialgeometrie der Kreise und Kugeln. Bearbeitet von G Thomsen* (Berlin: Springer)

[19] Musso E and Nicolodi L 1995 L-minimal canal surfaces *Rend. Mat. Appl., VII. Ser.* **15** 421–45

[20] Musso E and Nicolodi L 1999 On the equation defining isothermic surfaces in Laguerre geometry *New Developments in Differential Geometry Proc. Conf. (Budapest)* ed J Szenthe (Dordrecht: Kluwer) pp 285–94

[21] Musso E and Nicolodi L 1996 A variational problem for surfaces in Laguerre geometry *Trans. Am. Math. Soc.* **348** 4321–37

[22] Musso E and Nicolodi L 1997 Isothermal surfaces in Laguerre geometry *Boll. Unione Mat. Ital., VII. Ser.* B **11** (Suppl.) 125–44

[23] Musso E and Nicolodi L 1999 Laguerre geometry of surfaces with plane lines of curvature *Abh. Math. Semin. Univ. Hamb.* **69** 123–38

[24] Musso E and Nicolodi L 2000 The Bianchi–Darboux transform of L-isothermic surfaces *Int. J. Math.* **11** 911–24

[25] Musso E and Nicolodi L 2005 On the Cauchy problem for the integrable system of Lie minimal surfaces *J. Math. Phys.* **46** 113509–15

[26] Musso E and Nicolodi L 2006 Deformation and applicability of surfaces in Lie sphere geometry *Tohoku Math. J.* **58** 161–87

[27] Szereszewski A 2009 L-isothermic and L-minimal surfaces *J. Phys. A: Math. Theor.* **42** 115203

[28] Schief W K, Szereszewski A and Rogers C 2009 On shell membranes of Enneper type *J. Phys. A: Math. Theor.* **42** 404016

[29] Bianchi L 1915 Sulla generazione, per rotolamento, delle superficie isoterme e delle superficie a rappresentazione isoterma delle linee di curvatura *Rom. Acc. L. Rend.* **24** 377–87

[30] Eisenhart L P 1923 *Transformations of Surfaces* (Princeton, NJ: Princeton University Press)